Entry Task:

Complete the square on each of these:

half → 5 → square

$$x^2 + 10x = x^2 + 10x + 25 - 25 = (x+5)^2 - 25$$

So $\boxed{x^2 + 10x = (x+5)^2 - 25}$

half → $\frac{1}{2}$ → square

$$4x^2 + 4x + 3 = 4\left(x^2 + x + \tfrac{3}{4}\right) = 4\left(\underbrace{x^2 + x + \tfrac{1}{4}}_{(x+\frac{1}{2})^2} \underbrace{- \tfrac{1}{4} + \tfrac{3}{4}}_{\frac{1}{2}}\right) = 4\left((x+\tfrac{1}{2})^2 + \tfrac{1}{2}\right)$$

So $\boxed{4x^2 + 4x + 3 = 4\left((x+\tfrac{1}{2})^2 + \tfrac{1}{2}\right)}$

half → 4 → squared

$$14 - 8x - x^2 = 14 - (x^2 + 8x) = 14 - (x^2 + 8x + 16 - 16)$$
$$= 14 - ((x+4)^2 - 16)$$
$$= 14 - (x+4)^2 + 16$$
$$= 30 - (x+4)^2$$

So

$\boxed{14 - 8x - x^2 = 30 - (x+4)^2}$

7.3 Trigonometric Substitution Summary

CASE	SUBSTITUTION
$\sqrt{a^2 - u^2}$	$u = a\sin(\theta)$ $du = a\cos(\theta)\,d\theta$ $\sqrt{a^2 - a^2\sin^2(\theta)} = \mathrm{a}\cos(\theta)$
$\sqrt{a^2 + u^2}$	$u = a\tan(\theta)$ $du = a\sec^2(\theta)\,d\theta$ $\sqrt{a^2 + a^2\tan^2(\theta)} = \mathrm{a}\sec(\theta)$
$\sqrt{u^2 - a^2}$	$u = a\sec(\theta)$ $du = a\sec(\theta)\tan(\theta)\,d\theta$ $\sqrt{a^2\sec^2(\theta) - a^2} = \mathrm{a}\tan(\theta)$

1. Trig Sub
2. Trig Integral (use 7.2 methods)
3. Triangle Trick

f you encounter a `middle term'

$$\sqrt{ax^2 + \boldsymbol{bx} + c}.$$

Complete ~~the~~ square!

$$34 - 6x + x^2 = 34 + x^2 - 6x$$
$$= 34 + x^2 - 6x + 9 - 9$$
$$= 25 + (x-3)^2$$

CHECK!!!!! $25 + x^2 - 6x + 9$ ✓

Full Example:

$$\int \frac{x}{\sqrt{34 - 6x + x^2}}\,dx$$

1. Complete the square.
2. Do a trig sub problem.

$$\int \frac{x}{\sqrt{25 + (x-3)^2}}\,dx$$

$x - 3 = 5\tan(\theta)$

$x = 5\tan(\theta) + 3$

$dx = 5\sec^2(\theta)\,d\theta$

$$= \int \frac{(5\tan(\theta) + 3)\,5\sec^2\theta\,d\theta}{5\sec\theta}$$

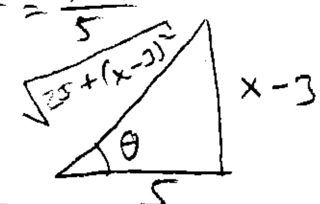
$$= \int (5\tan\theta + 3)\sec\theta\,d\theta$$

$$= \int 5\sec\theta\tan\theta + 3\sec\theta\,d\theta$$

$$= 5\sec\theta + 3\ln|\sec\theta + \tan\theta| + C$$

$$\uparrow$$
$$\frac{\sqrt{25 + (x-3)^2}}{5}$$

$\tan\theta = \frac{x-3}{5}$



$$= \boxed{\sqrt{25 + (x-3)^2} + 3\ln\left|\frac{\sqrt{25+(x-3)^2}}{5} + \frac{x-3}{5}\right| + C}$$

7.4 Partial Fractions

Motivation: We will learn to break-up fractions like:

$$\frac{x^3+4x-4}{x^2(x^2+4)} = \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^2+4}$$

Entry Task 1: If I tell you the above relationship is true, then integrate

$$\int \frac{x^3+4x-4}{x^2(x^2+4)}\,dx$$

$$= \int \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^2+4}\,dx$$

$$= \ln|x| + \frac{1}{x} + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

NOTE: $\int \frac{1}{x^2+4}\,dx = \frac{1}{4}\int \frac{1}{\frac{x^2}{4}+1}\,dx$

$$= \frac{1}{4}\int \frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx$$

$t = \frac{1}{2}x$

$dt = \frac{1}{2}dx$

$2dt = dx$

$$= \frac{1}{4}\int \frac{1}{t^2+1}\,2dt$$

$$= \frac{1}{2}\tan^{-1}(t) + C$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

Entry Task 2: Do you know **long-division**?

Divide in order to fill in the question marks:

$$\frac{576}{11} = ? + \frac{?}{11}$$

$$\begin{array}{r} 52 \\ 11\overline{)576} \\ -55 \\ \hline 26 \\ -22 \\ \hline 4 \end{array}$$

$$\Rightarrow \frac{576}{11} = 52 + \frac{4}{11}$$

Partial Fraction Decomposition

Step 0: Is the fraction *reduced*?

reduced - highest power on top smaller than the highest power on bottom.

If yes, move to step 1.

If not, divide, then move to step 1.

Example:

$$\int \frac{x^2+x}{x+3}\,dx$$

HIGHEST NUM. POWER = 2

HIGHEST DEN. POWER = 1

$2 \geq 1 \Rightarrow$ DIVIDE!

$$\begin{array}{r} x - 2 \\ x+3 \,\overline{\smash{)}\, x^2 + x} \\ -(x^2+3x) \\ \hline -2x \\ -(-2x-6) \\ \hline 6 \end{array}$$

$$\Rightarrow \frac{x^2+x}{x+3} = x-2+\frac{6}{x+3}$$

CHECK $\frac{(x-2)(x+3)}{x+3} + \frac{6}{x+3}$

$\frac{x^2+x-6+6}{x+3}$ ✓

$$= \int x-2+\frac{6}{x+3}\,dx$$

$$= \boxed{\frac{1}{2}x^2 - 2x + 6\ln|x+3| + C}$$

CHECK $\frac{d}{dx} \int$ $x-2+\frac{6}{x+3}$ ✓

Partial Fractions Method Summary

Step 0: Reduce (if needed), see last page.

Step 1: Factor Denominator.
Write out decomposition below:

i) Distinct Linear:

$$\frac{x^2-3}{x(x-1)(x+4)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+4}$$

ii) Repeated Linear:

$$\frac{5+2x}{(x+3)(x-2)^3} = \frac{A}{x+3} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3}$$

iii) Irreducible Quadratic:

$$\frac{4x}{(x+1)(x^2+9)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+9}$$

Step 2: Solve for A, B, C ....

Step 3: Integrate

All the integrals in this section look like these:

$$\int \frac{1}{2x+5}\,dx = \frac{1}{2}\ln|2x+5| + C$$

$$\int \frac{1}{(x-4)^2}\,dx = -\frac{1}{x-4} + C$$

$$\int \frac{1}{(x+7)^3}\,dx = -\frac{1}{2}\frac{1}{(x+7)^2} + C$$

$$\int \frac{1}{x^2+9}\,dx = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$$

$$\int \frac{x}{x^2+9}\,dx = \frac{1}{2}\ln|x^2+9| + C$$

The method uses algebra to rewrite ***any*** rational function as a sum of the integrals like those above.

Example:

$$\int \frac{x+1}{x^2-4}\,dx$$

TOP POWER = 1

BOT POWER = 2

$1 < 2 \Rightarrow$ ~~[illegible]~~ REDUCED ✓

DON'T NEED TO DIVIDE

$$\frac{x+1}{(x+2)(x-2)} \stackrel{?}{=} \frac{A}{x+2} + \frac{B}{x-2}$$

$$\Rightarrow \quad x+1 = A(x-2) + B(x+2)$$

$$\Rightarrow \quad x+1 = Ax - 2A + Bx + 2B$$

$$\underbrace{x+1} = \underbrace{(A+B)x + (-2A+2B)}$$

WANT ALWAYS THE SAME!

FACTOR!

$$\int \frac{1/4}{x+2} + \frac{3/4}{x-2}\,dx$$

$$\boxed{\frac{1}{4}\ln|x+2| + \frac{3}{4}\ln|x-2| + C}$$

MATCH COEFFICIENTS!

$\Rightarrow \quad A+B = 1 \Rightarrow B = 1-A$

and $-2A+2B = 1 \Rightarrow -2A + 2(1-A) = 1$

$\Rightarrow \quad -4A = -1 \Rightarrow A = 1/4$

$B = 1-A = 3/4$

SHORTCUT!

LET $x=2 \Rightarrow \quad 3 = A(2-2) + B(2+2)$

$\Rightarrow B = 3/4$

$x=-2 \Rightarrow \quad -1 = A(-2-2) + B(-2+2)$

$\Rightarrow A = 1/4$

Example:

$$\int \frac{x+1}{x^3+3x^2}\,dx$$

REDUCED ✓

FACTOR DENOM

$$\frac{x+1}{x^2(x+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+3}$$

REPEATED

$$\Rightarrow \quad x+1 = Ax(x+3) + B(x+3) + Cx^2$$

$$x=0 \Rightarrow 1 = A(0) + B(3) + C(0)$$

$$\Rightarrow \boxed{B = 1/3}$$

$$x=-3 \Rightarrow -2 = A(0) + B(0) + C(9)$$

$$\Rightarrow \boxed{C = -2/9}$$

$$x+1 = Ax^2 + 3Ax + Bx + 3B + Cx^2$$

$$x+1 = (A+C)x^2 + (3A+B)x + 3B$$

$$\Downarrow$$

$$A + C = 0 \Rightarrow A = -C$$

$$3A + B = 1$$

$$3B = 1$$

$$\boxed{A = -(-2/9) = 2/9}$$

$$\int \frac{x+1}{x^2(x+3)}\,dx$$

$$= \int \frac{2/9}{x} + \frac{1/3}{x^2} + \frac{-2/9}{x+3}\,dx$$

$$= \frac{2}{9}\ln|x| - \frac{1}{3}\frac{1}{x} - \frac{2}{9}\ln|x+3| + C$$

ASIDE:

$$\frac{2}{9}\left(\ln|x| - \ln|x+3|\right) = \frac{2}{9}\ln\left|\frac{x}{x+3}\right|$$

Example:

$$\int \frac{x^2 - x + 6}{x^3 + 3x} dx$$

REDUCED ✓

FACTOR DENOM

$$\frac{x^2 - x + 6}{x(x^2+3)} = \frac{A}{x} + \frac{Bx + C}{x^2+3}$$

$$\int \frac{x^2 - x + 6}{x(x^2+3)} dx$$

$$\Rightarrow x^2 - x + 6 = A(x^2+3) + (Bx+C)x$$

→ SPLIT UP!

$$x = 0 \Rightarrow 6 = A(3) + (B(0)+C)(0)$$

$$\Rightarrow A = 2$$

$$= \int \frac{2}{x} + \frac{-x-1}{x^2+3} dx$$

$$x^2 - x + 6 = Ax^2 + 3A + Bx^2 + Cx$$

$$\Rightarrow A + B = 1 \Rightarrow B = 1 - A = -1$$

$$C = -1$$

$$3A = 6 \checkmark$$

$$= \int \frac{2}{x} dx - \int \frac{x}{x^2+3} - \int \frac{1}{x^2+3} dx$$

$$= 2\ln|x| - \int \frac{x}{u} \frac{1}{2x} du - \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

$$= 2\ln|x| - \frac{1}{2}\ln|u| - \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

$$u = x^2 + 3$$

$$du = 2x\,dx$$

$$\frac{1}{2x} du = dx$$

$$= \boxed{2\ln|x| - \frac{1}{2}\ln(x^2+3) - \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C}$$

Example:

$$\int \frac{x}{x^2+4x+5}\,dx$$

REDUCED ✓

FACTOR DENOM???

$x^2+4x+5=0$ HAS

NO REAL SOL'NS

$x = \dfrac{-4 \pm \sqrt{16-4(5)}}{2}$

-4 under radical

IRREDUCIBLE

$\Downarrow$

COMPLETE THE SQUARE!

$$x^2+4x+\underbrace{4}_{}-4+5=(x+2)^2+1$$

HALF $\to$ 2 $\to$ SQUARED

to simplify

$t = x+2 \to x = t-2$

$dt = dx$

$$\int \frac{x}{(x+2)^2+1}\,dx$$

$$\int \frac{t-2}{t^2+1}\,dt$$

NOW SPLIT UP!

CE
AST
AGE

$$\int \frac{t}{t^2+1}\,dt - \int \frac{2}{t^2+1}\,dt$$

$$= \frac{1}{2}\ln(t^2+1) - 2\tan^{-1}(t) + C$$

$$= \boxed{\frac{1}{2}\ln\left((x+2)^2+1\right) - 2\tan^{-1}(x+2) + C}$$

How to integrate

A. Look for simplifications/substitutions

B. Products/Logs/Inverse Trig → BY PARTS
Sin/Cos/Tan/Sec combos → TRIG
Quadratic (under a radical) → TRIG SUB
Rational Function → PART. FRAC.

C. If nothing seems to work, substitution.
(u = inside, u = $\sqrt{\ }$, u = trig, u = e^x)

Examples of substitution:

1. $\int e^{\sqrt{x}}\,dx$

$t=\sqrt{x} \Rightarrow t^2 = x$
$2t\,dt = dx$

$\int e^t\, 2t\,dt$

$\int 2te^t\,dt$

NOW BY PARTS!

2. $\int \frac{3}{x-2\sqrt{x}}\,dx$

$t=\sqrt{x} \Rightarrow t^2=x$
$2t\,dt=dx$

$\int \frac{3}{t^2-2t}\,2t\,dt$

$= \int \frac{6t}{t^2-2t}\,dt$

NOW FACTOR DENOMINATOR!

3. $\int \frac{\cos(x)}{4-\sin^2(x)}\,dx$

$t=\sin(x)$
$dt=\cos(x)\,dx$

$\int \frac{1}{4-t^2}\,dt$

NOW FACTOR DENOMINATOR!

4. $\int e^x \cos(e^x)\sin^3(e^x)\,dx$

$t=e^x$
$dt=e^x\,dx$

$\int \cos(t)\sin^3(t)\,dt$

NOW "PULL-OUT" $\cos(t)$ AND $u=\sin(t)$!

How would you *start* these?

1. $\int \tan^3(x)\sec(x)\,dx = \int \tan^2(x)\sec(x)\tan(x)\,dx$ $\qquad u = \sec(x)$

TRIG!

$= \int (\sec^2(x)-1)\sec(x)\tan(x)\,dx$ $\qquad du = \sec(x)\tan(x)\,dx$

---

2. $\int x^2\ln(x)\,dx = \frac{1}{3}x^3\ln(x) - \int \frac{1}{3}x^2\,dx$ $\qquad u = \ln(x) \quad dv = x^2\,dx$

BY PARTS!

$= \frac{1}{3}x^3\ln(x) - \frac{1}{9}x^3 + C$ $\qquad du = \frac{1}{x}\,dx \quad v = \frac{1}{3}x^3$

---

3. $\int x\sqrt{5-x^2}\,dx = \int x\sqrt{u}\,\frac{1}{-2x}\,du = -\frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C$

SUB. (OR TRIG SUB)

$= -\frac{1}{3}(5-x^2)^{3/2} + C$

$u = 5 - x^2$ ← EASIER $\qquad x = \sqrt{5}\sin(\theta)$

$du = -2x\,dx$ $\qquad$ OR

$-\frac{1}{2x}\,du = dx$

---

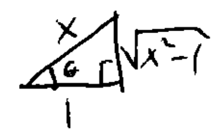
4. $\int \frac{\sqrt{x^2-1}}{x^2}\,dx = \int \frac{\sqrt{\sec^2\theta - 1}}{\sec^2\theta}\sec\theta\tan\theta\,d\theta = \int \frac{\tan^2\theta}{\sec\theta}\,d\theta$ OR

TRIG SUB

$\int \frac{\sec^2\theta - 1}{\sec\theta}\,d\theta = \int \frac{\sin^2\theta}{\cos\theta}\,d\theta$

$x = \sec\theta$

$dx = \sec\theta\tan\theta\,d\theta$



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5. $\int \frac{x^2+1}{x^2-2x-3}\,dx = \int 1 + \frac{2x+4}{x^2-2x-3}\,dx$

DIVIDE, THEN FACTOR DENOM

$\frac{2x+4}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$

$$\begin{array}{r|l} & 1 \\ x^2-2x-3 & x^2+1 \\ & -(x^2-2x-3) \\ & 2x+4 \end{array}$$

---

6. $\int x\tan^{-1}(x)\,dx = \frac{1}{2}x^2\tan^{-1}(x) - \frac{1}{2}\int \frac{x^2}{x^2+1}\,dx$ $\qquad u = \tan^{-1}(x) \quad dv = x\,dx$

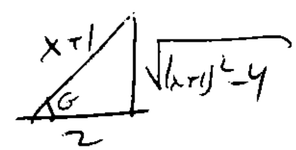
BY PARTS $\qquad$ ← DIVIDE!! $\qquad du = \frac{1}{x^2+1}\,dx \quad v = \frac{1}{2}x^2$

---

7. $\int \frac{dx}{\sqrt{4x^2+8x-12}}\,dx$

$\sqrt{4(x^2+2x+1-1-3)} = 2\sqrt{(x+1)^2-4}$

$x+1 = 2\sec\theta$



$\to \int \frac{1}{2\sqrt{4\sec^2\theta - 4}}\,2\sec\theta\tan\theta\,d\theta$

COMPLETE SQUARE & TRIG SUB!

$= \frac{1}{2}\int \frac{1}{\cancel{2}\cancel{\tan\theta}}\cancel{2}\sec\theta\cancel{\tan\theta}\,d\theta = \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$